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# On Consistency of the Quantum-Like Representation Algorithm

**Peter Nyman**

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**Abstract** In this paper we continue to study so-called "inverse Born's rule problem": to construct a representation of probabilistic data of any origin by a complex probability amplitude which matches Born's rule. The corresponding algorithm—quantum-like representation algorithm (QLRA)—was recently proposed by A. Khrennikov (Found. Phys. 35(10):1655–1693, 2005; Physica E 29:226–236, 2005; Dokl. Akad. Nauk 404(1):33–36, 2005; J. Math. Phys. 46(6):062111–062124, 2005; Europhys. Lett. 69(5):678–684, 2005). Formally QLRA depends on the order of conditioning. For two observables (of any origin, e.g., physical or biological) $a$ and $b$, $b|a$- and $a|b$ conditional probabilities produce two representations, say in Hilbert spaces $H^{b|a}$ and $H^{a|b}$. In this paper we prove that under "natural assumptions" (which hold, e.g., for quantum observables represented by operators with nondegenerate spectra) these two representations are unitary equivalent. This result proves the consistency of QLRA.

**Keywords** Quantum-like representation algorithm · Inverse Born's rule problem · Order of conditioning · Unitary equivalence of representations

## 1 Introduction

During the last 80 years tremendous efforts have been made to clarify the inter-relation between classical and quantum probabilities; see, e.g., von Neumann [1] for the first detailed presentation of this problem and see, e.g., Gudder [2–4], Svozil [5, 6], Fine [7], Garola et al. [8–10], Dvurecenskij and Pulmanova [12], Ballentine [11], O. Nánásiová et al. [13, 14], Allahverdyan et al. [15] for modern studies.[1] We remark that during the last

[1] The list of references is far from complete, see Khrennikov's monograph [25] for the detailed list of references.

P. Nyman (✉)
International Center for Mathematical Modelling in Physics and Cognitive Sciences,
University of Växjö, 35195 Växjö, Sweden
e-mail: peter.nyman@vxu.se

30 years the main interest was attracted to Bell's inequality; see, e.g. [16–19], for detailed a presentation. However, the basic rule of QM is Born's rule. Therefore the study of its origin is not less (and maybe even more) important than investigations of Bell's inequality. In this paper we continue to study so-called "inverse Born's rule problem" as it was formulated by Khrennikov [20–24]:

**IBP** (Inverse Born Problem) To construct a representation of probabilistic data by a complex probability amplitude which matches Born's rule.

The solution of IBP provides a possibility to represent probabilistic data by "wave functions" and operate with this data by using linear algebra (as we do in conventional QM). In a special case (for a pair of dichotomous observables) this problem was solved in [20–24] with the help of so-called quantum-like representation algorithm—QLRA.

Formally, the output of QLRA depends on the order of conditioning. For two observables $a$ and $b$, $b|a$- and $a|b$ conditional probabilities produce two representations, say in Hilbert spaces $H^{b|a}$ and $H^{a|b}$. In this paper we prove that under natural assumptions these two representations are unitary equivalent. This result proves the consistency of QLRA.

"Natural assumptions" about probabilities which are used in this paper hold, e.g., for probabilistic data obtained in measurements of *quantum observers with nondegenerate spectra*; see Appendix.

We want to discuss IBP in detail. Consider probabilistic data collected in measurements of a pair of observables, say $a$ and $b$. These observables need not be quantum observables. They can describe measurement done in any domain of science. We have implemented a representation of such data by a complex amplitude or (in the abstract Hilbert space formalism) by the normalized vector in a complex Hilbert space. It provides us with a possibility of applying linear algebra to an operation with probabilities (through an operation with probability amplitudes). Clearly, such representation of data is not always possible to find. We should find constraints for constructing a representation of these data in a complex Hilbert space. We show that under natural assumptions one can introduce an algorithmic procedure of representation of probabilistic data by complex probability amplitudes matching Born's rule. In quantum physics our algorithm, QLRA, just reproduces the wave function of a quantum system (in special cases of observables represented by self-adjoint operators with nondegenerate spectra; see Appendix); so, it is a quantum state reconstruction algorithm. However, QLRA can be applied for data collected in any domain of science; see, e.g., [26–28], to applications to cognitive science and psychology.

## 2 Inversion of Born's Rule

We consider the simplest situation. There are given two dichotomous observables of any origin: $a = \alpha_1, \alpha_2$ and $b = \beta_1, \beta_2$. We set $X_a = \{\alpha_1, \alpha_2\}$ and $X_b = \{\beta_1, \beta_2\}$—"spectra of observables".

We assume that there is given the matrix of transition probabilities $\mathbf{P}^{b|a} = (p^{b|a}_{\beta\alpha})$, where $p^{b|a}_{\beta\alpha} \equiv P(b = \beta|a = \alpha)$ is the probability to obtain the result $b = \beta$ under the condition that the result $a = \alpha$ has been obtained. There are also given probabilities $p^a_\alpha \equiv P(a = \alpha), \alpha \in X_a$, and $p^b_\beta \equiv P(b = \beta), \beta \in X_b$. Probabilistic data $C = \{p^a_\alpha, p^b_\beta\}$ are related to some experimental context (in the physics preparation procedure).

IBP is to represent this data by a probability amplitude $\psi$ (in the simplest case it is complex-valued) such that Born's rule holds for both observables:

$$p_\beta^b = |\langle \psi, e_\beta^{b|a} \rangle|^2, \qquad p_\alpha^a = |\langle \psi, e_\alpha^{b|a} \rangle|^2, \tag{1}$$

where $\{e_\beta^{b|a}\}_{\beta \in X_b}$ and $\{e_\alpha^{b|a}\}_{\alpha \in X_a}$ are orthonormal bases for observables $b$ and $a$, respectively (so the observables are diagonal in the respective bases).

In [20–24] the solution of IBP is given in the form of an algorithm which constructs a probability amplitude from the data. Formally, the output of this algorithm depends on the order of conditioning. By starting with the matrix of transition probabilities $\mathbf{P}^{a|b}$, instead of $\mathbf{P}^{b|a}$, we construct another probability amplitude $\psi^{a|b}$ (the amplitude in (1) should be denoted by $\psi^{b|a}$) and other bases, $\{e_\beta^{a|b}\}_{\beta \in X_b}$ and $\{e_\alpha^{a|b}\}_{\alpha \in X_a}$. We shall see that under natural assumptions these two representations are unitary equivalent.

## 3 QLRA

### 3.1 $H^{b|a}$-Conditioning

Suppose that the matrix of transition probabilities $\mathbf{P}^{b|a}$ is given. In [20–24] the following formula for the interference of probabilities (generalizing the classical formula of total probability) was derived: $p_\beta^b = \sum_\alpha p_\alpha^a p_{\beta\alpha}^{b|a} + 2\lambda_\beta \sqrt{\prod_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}}$, where the "coefficient of interference" is given by

$$\lambda_\beta = \frac{p_\beta^b - \sum_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}}{2\sqrt{\prod_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}}}. \tag{2}$$

We shall proceed under the conditions:

(1) $\mathbf{P}^{b|a}$ is doubly stochastic.
(2) Probabilistic data $C = \{p_\alpha^a, p_\beta^b\}$ consist of strictly positive probabilities.
(3) The coefficients of interference $\lambda_\beta$, $\beta \in X_b$, are bounded by one: $|\lambda_\beta| \leq 1$.

Probabilistic data $C$ such that (3) holds are called *trigonometric*, because in this case we have the conventional formula of trigonometric interference:[2] $p_\beta^b = \sum_\alpha p_\alpha^a p_{\beta\alpha}^{b|a} + 2\cos\theta_\beta \sqrt{\prod_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}}$, where

$$\lambda_\beta = \cos\theta_\beta. \tag{3}$$

By using the elementary formula: $D = A + B + 2\sqrt{AB}\cos\theta = |\sqrt{A} + e^{i\theta}\sqrt{B}|^2$, for real numbers $A, B > 0, \theta \in [0, 2\pi]$, we can represent the probability $p_\beta^b$ as the square of the complex amplitude (Born's rule): $p_\beta^b = |\psi_\beta^{b|a}|^2$. Here

$$\psi_\beta^{b|a} = \sqrt{p_{\alpha_1}^a p_{\beta\alpha_1}^{b|a}} + e^{i\theta_\beta} \sqrt{p_{\alpha_2}^a p_{\beta\alpha_2}^{b|a}}, \quad \beta \in X_b. \tag{4}$$

[2] This formula can be easily derived in the conventional QM formalism; see, e.g., [25], by transition from the basis of eigenvectors for the $a$-observable to the basis of eigenvectors for the $b$-observables. We recall that in QM observables are given by self-adjoint operators. However, we proceed in the opposite direction. We would like to produce a complex probability amplitude and operator representation of the observables by using this formula.

The formula (4) gives the quantum-like representation algorithm—QLRA. For any trigonometric probabilistic data $C$, QLRA produces the complex amplitude $\psi^{b|a}$ (the normalized vector in the two dimensional complex Hilbert space, say $H^{b|a}$):

$$\psi^{b|a} = \psi_{\beta_1}^{b|a} e_{\beta_1}^{b|a} + \psi_{\beta_2}^{b|a} e_{\beta_2}^{b|a}, \tag{5}$$

where

$$e_{\beta_1}^{b|a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_{\beta_2}^{b|a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To solve IBP completely, we would like to have Born's rule not only for the $b$-variable, but also for the $a$-variable: $p_\alpha^a = |\langle \psi^{b|a}, e_\alpha^{b|a} \rangle|^2, \alpha \in X_a$. Here the $a$-basis in the Hilbert space $H^{b|a}$ is given, see [20–24] for details, by

$$e_{\alpha_1}^{b|a} = \begin{pmatrix} \sqrt{p_{\beta_1\alpha_1}^{b|a}} \\ \sqrt{p_{\beta_2\alpha_1}^{b|a}} \end{pmatrix}, \qquad e_{\alpha_2}^{b|a} = \begin{pmatrix} \sqrt{p_{\beta_1\alpha_2}^{b|a}} \\ -\sqrt{p_{\beta_2\alpha_2}^{b|a}} \end{pmatrix}.$$

It is orthonormal, since $\mathbf{P}^{b|a}$ is assumed to be doubly stochastic. In this basis the amplitude $\psi^{b|a}$ is represented as

$$\psi^{b|a} = \sqrt{p_{\alpha_1}^a} e_{\alpha_1}^{b|a} + e^{i\theta_{\beta_1}} \sqrt{p_{\alpha_2}^a} e_{\alpha_2}^{b|a}. \tag{6}$$

We recall that in QM a pure state $\Psi$ is defined as an equivalent class with respect to multipliers of the form $c = e^{i\gamma}$. We shall use a similar terminology. Each complex amplitude $\psi^{b|a}$ produced by QLRA determines a *quantum-like state* (representing given probabilistic data)—the equivalence class $\Psi^{b|a}$ being determined by the representative $\psi^{b|a}$.

### 3.2 $H^{a|b}$-Conditioning

Here

$$\psi_\alpha^{a|b} = \sqrt{p_{\beta_1}^b p_{\alpha\beta_1}^{a|b}} + e^{i\theta_\alpha} \sqrt{p_{\beta_2}^b p_{\alpha\beta_2}^{a|b}}, \quad \alpha \in X_a. \tag{7}$$

For any trigonometric probabilistic data $C$, QLRA produces the complex amplitude $\psi^{a|b}$ (the normalized vector in the two dimensional complex Hilbert space, say $H^{a|b}$) :

$$\psi^{a|b} = \psi_{\alpha_1}^{a|b} e_{\alpha_1}^{a|b} + \psi_{\alpha_2}^{a|b} e_{\alpha_2}^{a|b}, \tag{8}$$

where

$$e_{\alpha_1}^{a|b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_{\alpha_2}^{a|b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here the $b$-basis in the Hilbert space $H^{a|b}$ is given by

$$e_{\beta_1}^{a|b} = \begin{pmatrix} \sqrt{p_{\alpha_1\beta_1}^{a|b}} \\ \sqrt{p_{\alpha_2\beta_1}^{a|b}} \end{pmatrix}, \qquad e_{\beta_2}^{a|b} = \begin{pmatrix} \sqrt{p_{\alpha_1\beta_2}^{a|b}} \\ -\sqrt{p_{\alpha_2\beta_2}^{b|a}} \end{pmatrix}.$$

In this basis the amplitude $\psi^{a|b}$ is represented as

$$\psi^{a|b} = \sqrt{p_{\beta_1}^b} e_{\beta_1}^{a|b} + e^{i\theta_{\alpha_1}} \sqrt{p_{\beta_2}^b} e_{\beta_2}^{b|a}. \tag{9}$$

As in the case of $H^{b|a}$-representation, the *quantum-like state* (representing given probabilistic data) is defined as the equivalence class $\Psi^{a|b}$ with the representative $\psi^{a|a}$.

## 4 Unitary Equivalence of $b|a$- and $a|b$-Representations

Thus, as we have seen by selecting two types of conditioning, we represented the probabilistic data $C = \{p^a_\alpha, p^b_\beta\}$ by two quantum-like states, $\Psi^{b|a}$ and $\Psi^{a|b}$. We are interested in the consistency of these representations.

We remark that any linear operator $W : H^{b|a} \to H^{a|b}$ induces the map of equivalence classes of the unit spheres with respect to multipliers $c = e^{i\gamma}$. We define the unitary operator $U^{a|b}_{b|a} : H^{b|a} \to H^{a|b}$ by $U(e^{b|a}_\alpha) = e^{a|b}_\alpha, \alpha \in X_a$. It induces the mentioned map of equivalent classes.

**Theorem** *The operator $U^{a|b}_{b|a}$ maps $\Psi^{b|a}$ into $\Psi^{a|b}$ if and only if the following inter-relation of symmetry takes place for the matrices of transition probabilities $\mathbf{P}^{b|a}$ and $\mathbf{P}^{a|b}$:*

$$p^{b|a}_{\beta\alpha} = p^{a|b}_{\alpha\beta}, \tag{10}$$

*for all $\alpha$ and $\beta$ from the spectra of observables $a$ and $b$.*

*Proof* Take the representative of $\Psi^{b|a}$ given by (6). Then

$$U^{a|b}_{b|a}\psi^{b|a} = \sqrt{p^a_{\alpha_1}}e^{a|b}_{\alpha_1} + e^{i\theta_{\beta_1}}\sqrt{p^a_{\alpha_2}}e^{a|b}_{\alpha_2}. \tag{11}$$

Our aim is to show that this vector is equivalent to the vector $\psi^{a|b}$ given by (8). By using $H^{b|a}$ analogs of (2) and (3) for the coefficients of interference and its cos-expression we determine $\cos\theta_{\alpha_1}$:

$$\begin{aligned} p^a_{\alpha_1} &= p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1} + p^b_{\beta_2}p^{a|b}_{\alpha_1\beta_2} + 2\cos\theta_{\alpha_1}\sqrt{p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1}p^b_{\beta_2}p^{a|b}_{\alpha_1\beta_2}} \\ &\Leftrightarrow \\ \cos\theta_{\alpha_1} &= \frac{p^a_{\alpha_1} - p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1} - p^b_{\beta_2}p^{a|b}_{\alpha_1\beta_2}}{2\sqrt{p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1}p^b_{\beta_2}p^{a|b}_{\alpha_1\beta_2}}}. \end{aligned} \tag{12}$$

We also calculate

$$\begin{aligned} \psi^{a|b}_{\alpha_2}\overline{\psi^{a|b}_{\alpha_1}} = p^b_{\beta_1}\sqrt{p^{a|b}_{\alpha_1\beta_1}p^{a|b}_{\alpha_2\beta_1}} - p^b_{\beta_2}\sqrt{p^{a|b}_{\alpha_2\beta_2}p^{a|b}_{\alpha_1\beta_2}} \\ - (\cos\theta_{\alpha_1} + i\sin\theta_{\alpha_1})\sqrt{p^b_{\beta_2}p^{a|b}_{\alpha_2\beta_2}p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1}} \\ + (\cos\theta_{\alpha_1} - i\sin\theta_{\alpha_1})\sqrt{p^b_{\beta_1}p^{a|b}_{\alpha_2\beta_1}p^b_{\beta_2}p^{a|b}_{\alpha_1\beta_2}}, \end{aligned} \tag{13}$$

where $\psi^{a|b}_{\alpha_2} = \sqrt{p^b_{\beta_1}p^{a|b}_{\alpha_2\beta_1}} - e^{i\theta_{\alpha_1}}\sqrt{p^b_{\beta_2}p^{a|b}_{\alpha_2\beta_2}}$ is given by (9). We use that $|\langle\psi^{a|b}_{\alpha_j}\rangle|^2 = p^a_{\alpha_j} \Leftrightarrow \psi^{a|b}_{\alpha_j} = \sqrt{p^a_{\alpha_j}}(\cos\gamma_{\alpha_j} + i\sin\gamma_{\alpha_j})$ where $\gamma_{\alpha_j} = \arg\psi^a_{\alpha_j}, j \in \{1, 2\}$ and this gives that

$$\psi^{a|b}_{\alpha_2}\overline{\psi^{a|b}_{\alpha_1}} = \sqrt{p^a_{\alpha_1}p^a_{\alpha_2}}(\cos(\gamma_{\alpha_2} - \gamma_{\alpha_1}) + i\sin(\gamma_{\alpha_2} - \gamma_{\alpha_1})). \tag{14}$$

The real part of (13) and (14) gives

$$\begin{aligned}
&\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cos(\gamma_{\alpha_2}-\gamma_{\alpha_1})\\
&\quad = p^b_{\beta_1}\sqrt{p^{a|b}_{\alpha_1\beta_1} p^{a|b}_{\alpha_2\beta_1}} - p^b_{\beta_2}\sqrt{p^{a|b}_{\alpha_2\beta_2} p^{a|b}_{\alpha_1\beta_2}}\\
&\qquad - \cos\theta_{\alpha_1}\Big(\sqrt{p^b_{\beta_2} p^{a|b}_{\alpha_2\beta_2} p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}} + \sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_2\beta_1} p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}}\Big).
\end{aligned} \tag{15}$$

Moreover, since $p^b_{\beta_2} = 1 - p^b_{\beta_1}$ and from the condition that $\mathbf{P}^{a|b}$ is double stochastic i.e. $p^{a|b}_{\alpha_1\beta_2} = p^{a|b}_{\alpha_2\beta_1} = 1 - p^{a|b}_{\alpha_1\beta_1} = 1 - p^{a|b}_{\alpha_2\beta_2}$, we rewrite (15)

$$\begin{aligned}
\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cos(\gamma_{\alpha_2}-\gamma_{\alpha_1}) &= (2p^b_{\beta_1}-1)\sqrt{p^{a|b}_{\alpha_1\beta_1}(1-p^{a|b}_{\alpha_1\beta_1})}\\
&\quad + \cos\theta_{\alpha_1}(1-2p^{a|b}_{\alpha_1\beta_1})\sqrt{(1-p^b_{\beta_1})p^b_{\beta_1}}.
\end{aligned} \tag{16}$$

Then by (2) and (3) we obtain $\cos\theta_{\beta_1}$:

$$\cos\theta_{\beta_1} = \frac{p^b_{\beta_1} - p^a_{\alpha_1} p^{b|a}_{\beta_1\alpha_1} - p^a_{\alpha_2} p^{b|a}_{\beta_1\alpha_2}}{2\sqrt{p^a_{\alpha_1} p^{b|a}_{\beta_1\alpha_1} p^a_{\alpha_2} p^{b|a}_{\beta_1\alpha_2}}}. \tag{17}$$

Multiply (17) with $2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}$ and use again that $p^b_{\beta_2} = 1 - p^b_{\beta_1}$ and $\mathbf{P}^{a|b}$ is double stochastic and

$$2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cos\theta_{\beta_1} = \frac{p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{b|a}_{\alpha_1\beta_1} - 2p^{b|a}_{\alpha_1\beta_1} p^a_{\alpha_1}}{\sqrt{p^{b|a}_{\alpha_1\beta_1} p^{b|a}_{\beta_1\alpha_2}}}. \tag{18}$$

We will show that $\cos(\gamma_{\alpha_2}-\gamma_{\alpha_1}) = \cos\theta_{\beta_1}$ or equivalent, we show that

$$2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cos(\gamma_{\alpha_2}-\gamma_{\alpha_1}) = 2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cos\theta_{\beta_1}. \tag{19}$$

Multiply $\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cos(\gamma_{\alpha_2}-\gamma_{\alpha_1})$ by $2\sqrt{p^{a|b}_{\alpha_1\beta_1}(1-p^{a|b}_{\alpha_1\beta_1})}$ on the left-hand side (16) such that $LHS = 2\sqrt{p^{a|b}_{\alpha_1\beta_1}(1-p^{a|b}_{\alpha_1\beta_1})}\ \sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\ \cos(\gamma_{\alpha_2}-\gamma_{\alpha_1})$ and replace $\cos\theta_{\alpha_1}$ with $\frac{p^a_{\alpha_1} - p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} - (1-p^b_{\beta_1})(1-p^{a|b}_{\alpha_1\beta_1})}{2\sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}}}$ on right-hand side

$$\begin{aligned}
LHS &= 2(2p^b_{\beta_1}-1)p^{a|b}_{\alpha_1\beta_1}(1-p^{a|b}_{\alpha_1\beta_1})\\
&\quad + (p^a_{\alpha_1} - p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} - (1-p^b_{\beta_1})(1-p^{a|b}_{\alpha_1\beta_1}))(1-2p^{a|b}_{\alpha_1\beta_1})\\
&= 2(2p^b_{\beta_1}-1)p^{a|b}_{\alpha_1\beta_1}(1-p^{a|b}_{\alpha_1\beta_1})\\
&\quad + (p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1})(1-2p^{a|b}_{\alpha_1\beta_1})\\
&= 2(2p^b_{\beta_1}-1)p^{a|b}_{\alpha_1\beta_1}(1-p^{a|b}_{\alpha_1\beta_1})\\
&\quad + (p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^{a|b}_{\alpha_1\beta_1} p^a_{\alpha_1})\\
&\quad - 2p^{a|b}_{\alpha_1\beta_1}(-1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}) - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}
\end{aligned}$$

$$
\begin{aligned}
&= 2(2p^b_{\beta_1} - 1)p^{a|b}_{\alpha_1\beta_1}(1 - p^{a|b}_{\alpha_1\beta_1}) \\
&\quad + (p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^{a|b}_{\alpha_1\beta_1} p^a_{\alpha_1}) \\
&\quad - 2p^{a|b}_{\alpha_1\beta_1}(-1 + 2p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}) \\
&= p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^{a|b}_{\alpha_1\beta_1} p^a_{\alpha_1}. \qquad (20)
\end{aligned}
$$

From (18) and (20) it will follow that

$$
\frac{p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{b|a}_{\alpha_1\beta_1} - 2p^{b|a}_{\alpha_1\beta_1} p^a_{\alpha_1}}{\sqrt{p^{b|a}_{\alpha_1\beta_1} p^{b|a}_{\beta_1\alpha_2}}} = \frac{p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^{a|b}_{\alpha_1\beta_1} p^a_{\alpha_1}}{\sqrt{p^{a|b}_{\alpha_1\beta_1} p^{a|b}_{\beta_1\alpha_2}}}
$$

$$\Leftrightarrow$$

$$
p^{b|a}_{\alpha_1\beta_1} = p^{a|b}_{\alpha_1\beta_1}. \qquad (21)
$$

Therefore we will conclude that $\cos(\gamma_{\alpha_2} - \gamma_{\alpha_1}) = \cos\theta_{\beta_1}$ iff $\mathbf{P}^{b|a} = \mathbf{P}^{a|b}$. Let

$$
U^{a|b}_{b|a} = \begin{pmatrix} \sqrt{p^{b|a}_{\beta_1\alpha_1}} & \sqrt{p^{b|a}_{\beta_1\alpha_2}} \\ \sqrt{p^{b|a}_{\beta_2\alpha_1}} & -\sqrt{p^{b|a}_{\beta_2\alpha_2}} \end{pmatrix}. \qquad (22)
$$

Then let us show that this vector is equivalent to the vector $\psi^{a|b}$ given by (8)

$$
\begin{aligned}
U^{a|b}_{b|a}\psi^{b|a} &= \sqrt{p^a_{\alpha_1}} e^{a|b}_{\alpha_1} + e^{i\theta_{\beta_1}} \sqrt{p^a_{\alpha_2}} e^{a|b}_{\alpha_2} \\
&= \sqrt{p^a_{\alpha_1}} e^{a|b}_{\alpha_1} + e^{i(\gamma_{\alpha_2} - \gamma_{\alpha_1})} \sqrt{p^a_{\alpha_2}} e^{a|b}_{\alpha_2}. \qquad (23)
\end{aligned}
$$

Then put $\psi^{a|b}_{\alpha_j} = \sqrt{p^a_{\alpha_j}} e^{i(\gamma_{\alpha_j})}$, $j \in \{1, 2\}$ into (8)

$$
\begin{aligned}
\psi^{a|b} &= \sqrt{p^a_{\alpha_1}} e^{i(\gamma_{\alpha_1})} e^{a|b}_{\alpha_1} + \sqrt{p^a_{\alpha_2}} e^{i(\gamma_{\alpha_2})} e^{a|b}_{\alpha_2} \\
&= e^{i(\gamma_{\alpha_1})} U^{a|b}_{b|a} \psi^{b|a}. \qquad (24)
\end{aligned}
$$

The complex amplitudes $\psi^{a|b}$ and $U^{a|b}_{b|a}\psi^{b|a}$ differ only by the multiplicative factor $e^{i(\gamma_{\alpha_1})}$. Hence, they belong to the same equivalent class of vectors on the unit sphere. Thus they are two representatives of the same quantum state $\Psi^{b|a}$. □

## Appendix: The Inter-Relation of Symmetry for Matrices of Transition Probabilities in the Case of Observables with Nondegenerate Spectra

We start with the general case of observables, which can have degenerate spectra. Let $Q_\beta$ and $P_\alpha$ be two orthogonal projection operators, then by the projection postulate of QM (or simply by definition of quantum conditional probability):

$$
p^{a|b}_{\alpha\beta} = \frac{\langle\psi|Q_\beta P_\alpha Q_\beta|\psi\rangle}{\langle\psi|Q_\beta|\psi\rangle} \qquad (25)
$$

and

$$p^{b|a}_{\beta\alpha} = \frac{\langle\psi|P_\alpha Q_\beta P_\alpha|\psi\rangle}{\langle\psi|P_\alpha|\psi\rangle} \tag{26}$$

hence, generally in the case of degenerate spectra, $p^{a|b}_{\alpha\beta} \neq p^{b|a}_{\beta\alpha}$. It contradicts (10).

Now let us consider the case of nondegenerate spectra. In this case we can write two orthogonal projection operators as $P_\alpha = |e_\alpha\rangle\langle e_\alpha|$ and $Q_\beta = |e_\beta\rangle\langle e_\beta|$. Hence,

$$\begin{aligned}
p^{a|b}_{\alpha\beta} &= \frac{\langle\psi|Q_\beta P_\alpha Q_\beta|\psi\rangle}{\langle\psi|Q_\beta|\psi\rangle} \\
&= \frac{\langle\psi|e_\beta\rangle\langle e_\beta|e_\alpha\rangle\langle e_\alpha|e_\beta\rangle\langle e_\beta|\psi\rangle}{\langle\psi|Q_\beta|\psi\rangle} \\
&= \frac{\langle\psi|e_\beta\rangle\langle e_\beta|\psi\rangle|\langle e_\alpha|e_\beta\rangle|^2}{\langle\psi|Q_\beta|\psi\rangle} \\
&= |\langle e_\alpha|e_\beta\rangle|^2
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
p^{b|a}_{\beta\alpha} &= \frac{\langle\psi|P_\alpha Q_\beta P_\alpha|\psi\rangle}{\langle\psi|P_\alpha|\psi\rangle} \\
&= \frac{\langle\psi|e_\alpha\rangle\langle e_\alpha|e_\beta\rangle\langle e_\beta|e_\alpha\rangle\langle e_\alpha|\psi\rangle}{\langle\psi|P_\alpha|\psi\rangle} \\
&= \frac{\langle\psi|e_\beta\rangle\langle e_\beta|\psi\rangle|\langle e_\beta|e_\alpha\rangle|^2}{\langle\psi|P_\alpha|\psi\rangle} \\
&= |\langle e_\beta|e_\alpha\rangle|^2
\end{aligned} \tag{28}$$

where $|\langle e_\alpha|e_\beta\rangle|^2 = |\langle e_\beta|e_\alpha\rangle|^2$. Finally, we obtain $p^{a|b}_{\alpha\beta} = p^{b|a}_{\beta\alpha}$ in this case.

We remark that this inter-relation of symmetry implies that both matrices of transition probabilities are *doubly stochastic*.

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